Supplement - Day 2

<u>Ex</u>: let $g(x) = x^3 e^x$

Evaluate $g'(x)$ at $x = 1$.

*product rule

$$g'(x) = (3x^2)(e^x) + (x^3)(e^x)$$

$$g'(1) = (3 \cdot 1^2)(e^1) + (1^3)(e^1)$$

$$= 3e + 1e = \boxed{4e}$$

Derivatives

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x} \quad \text{OR} \quad (\ln(x))' = \frac{1}{x}$$

$$\frac{d}{dx}(\ln(g(x))) = \frac{1}{g(x)} \cdot \frac{d}{dx}(g(x))$$

$$\text{OR} \quad (\ln(g(x)))' = \frac{g'(x)}{g(x)}$$

Ex: let $y = \ln(e^x)$, find y'

$$y' = \frac{1}{e^x}(e^x)' = \frac{1}{e^x} \cdot e^x = \frac{e^x}{e^x} = 1$$

Ex: let $f(x) = x\ln(x)$ find $f'(x)$.

*product rule

$$f'(x) = (x)'\ln(x) + (x)(\ln(x))'$$
$$= 1 \cdot \ln(x) + x \cdot \frac{1}{x}$$
$$= \ln(x) + 1$$

Ex: let $g(x) = \ln(\underline{\ln(\ln(\ln(x)))})$ find $g'(x)$.

* chain rule .... repeatedly!

$$g'(x) = \frac{1}{\ln(\ln(\ln(x)))} \cdot \left(\ln(\underline{\ln(\ln(x))})\right)'$$

$$= \frac{1}{\ln(\ln(\ln(x)))} \cdot \frac{1}{\ln(\ln(x))} \cdot \left(\ln(\underline{\ln(x)})\right)'$$

$$= \frac{1}{\ln(\ln(\ln(x)))} \cdot \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \left(\ln(x)\right)'$$

$$= \frac{1}{\ln(\ln(\ln(x)))} \cdot \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

Exponential Growth & Decay

We say $Q(t)$ grows exponentially as a function of time if

$$Q(t) = Q_0 e^{rt}$$

where Q_0 = quantity at time $t=0$

r = constant that depends on the problem

t = time

Consider the derivative.

$$Q'(t) = (Q_0)(e^{rt})(r)$$

$$= r(Q_0 e^{rt})$$

$$= r\,Q(t)$$

Ex: the graph of a function $g(x)$ passes through the point $(0, 7)$. Suppose that the slope of the tangent line to the graph of $y = g(x)$ at any point P is 3 times the y-coordinate of P. find $g(5)$.

* exponential growth

$$g'(P) = 3\,g(P) \rightarrow r = 3$$

Point $(0, 7)$ $\rightarrow Q_0 = 7$

then $g(x) = Q_0 e^{rt}$
$= 7e^{3t}$

so $g(5) = 7e^{3 \cdot 5} = 7e^{15}$

Continuously Compounded Interest

is calculated by

$$P(t) = P_0 e^{rt}$$

where $P(t)$ = principal after t years

P_0 = initial principal

r = interest rate per year

t = number of years.

Ex: if \$10,000 is invested at 3%, find the value of the investment after 7 years if the interest is compounded continuously.

$$P(t) = P_0 e^{rt}$$

$$= (10{,}000)e^{(.03)(7)}$$

$$= 10{,}000\, e^{.21}$$

$$\approx \$12{,}336.78$$

Ex: How many years will it take for an investment to double in value if the interest is compounded continuously at a rate of 5%?

$$\frac{2P_0}{P_0} = \frac{P_0 e^{.05t}}{P_0}$$

$$2 = e^{.05t}$$

$$\ln(2) = \ln(e^{.05t})$$

$$\frac{\ln 2}{.05} = \frac{.05t}{.05}$$

$$t = \frac{\ln 2}{.05} \approx \underline{13.86 \text{ years}}$$

Radioactive Decay Model

if Q_0 is the initial quantity of a radioactive substance with half-life t_0 then the quantity $Q(t)$ remaining at time t is modeled by $Q(t) = Q_0 e^{-rt}$

where $r = \dfrac{\ln 2}{t_0}$

Ex: The half life of Cesium-137 is 30 years. Suppose we have a 100 gram sample. How much of the sample will remain after 50 yrs?

$t_0 = 30 \qquad Q_0 = 100$

$$Q(t) = 100 e^{-\left(\frac{\ln 2}{30}\right)50}$$

$$= 100 \cancel{e}^{\cancel{\ln} 2\left(-\frac{50}{30}\right)}$$

$$= 100 \cdot 2^{-50/30} \approx 31.498 \text{ grams}$$